STICHTING
MATHEMATISCH CENTRUM
2e BOERHAAVESTRAAT 49
AMSTERDAM

S 254 (M 80 a)

Conditional limit-distributions for the entries in a 2×2-table.

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1960

Conditional limit-distributions for the entries in a 2×2-table *)

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UDC 519.2

Samenvatting

Voorwaardelijke limietverdelingen voor de stochastische grootheden in een 2×2-tabel.

In dit artikel worden de voorwaardelijke limietverdelingen van de stochastische grootheden in een 2 × 2-tabel beschouwd. Tevens wordt aangegeven hoe deze limietstellingen gebruikt kunnen worden om overschrijdingskansen bij een 2 × 2-tabel te benaderen.

1. Introduction

In this paper the conditional limit-distributions for the entries in a 2 × 2-table will be considered.

A 2 × 2-table e.g. occurs in the following situations:

1. Suppose an urn contains r white balls and s black balls; m balls are drawn at random without replacement. If $\underline{a}_1$[1]) is the number of white balls in the sample, the observations may be summarized in a 2 × 2-table as follows:

TABLE 1

	white	black	total
in the sample	$\underline{a}_1$	$\underline{a}_3$	m
not in the sample	$\underline{a}_2$	$\underline{a}_4$	n
total	r	s	N

In this table we have

$$(1;1) \qquad \underline{a}_2 = r - \underline{a}_1,\ \underline{a}_3 = m - \underline{a}_1,\ \underline{a}_4 = n - r + \underline{a}_1$$

and each of the random variables $\underline{a}_i$ has a hypergeometric distribution. For $\underline{a}_1$ e.g. we have

$$(1;2) \qquad \mathrm{P}\,[\underline{a}_1 = a] = \frac{\binom{m}{a}\binom{n}{r-a}}{\binom{N}{r}} = \frac{\binom{r}{a}\binom{s}{m-a}}{\binom{N}{m}}.$$

*) Report S 254 (M80a) of the Statistical Department of the Mathematical Centre, Amsterdam.

1) Random variables are distinguished from numbers (e.g. from the values they take in an experiment) by underlining their symbols.

2. Let the independent random variables $\underline{a}_1$ and $\underline{a}_2$ have binomial probability distributions, $\underline{a}_1$ with parameters m and p, $\underline{a}_2$ with parameters n and p. Then, under the condition $\underline{a}_1 + \underline{a}_2 = r$, the random variables $\underline{a}_i$ have hypergeometric distributions.

3. Let each of the independent random variables $\underline{x}$ and $\underline{y}$ assume only two values, e.g. 0 and 1. Let further N independent observations $(\underline{x}_j, \underline{y}_j)$ consist of $\underline{a}_1$ times (0,0), $\underline{a}_2$ times (0,1), $\underline{a}_3$ times (1,0) and $\underline{a}_4$ times (1,1). Then, under the conditions $\underline{a}_1 + \underline{a}_2 = r$ and $\underline{a}_1 + \underline{a}_3 = m$, the variables $\underline{a}_i$ have hypergeometric distributions.

In the second and third case the observations may also be summarized in a 2×2-table.

The exact tailprobability of a 2×2-table with fixed marginal totals may be obtained from the hypergeometric distribution (cf. R. A. Fisher (1948) and C. van Eeden (1953)). For the upper-tailprobability of $\underline{a}_1$ e.g. we have (cf. (1;1) and (1;2))

$$(1;3) \qquad \mathrm{P}\,[\underline{a}_1 \geqq a] = \mathrm{P}\,[\underline{a}_4 \geqq n - r + a] = \mathrm{P}\,[\underline{a}_2 \leqq r - a] = \\ = \mathrm{P}\,[\underline{a}_3 \leqq m - a] = \sum_{j \geqq a} \frac{\binom{m}{j}\binom{n}{r-j}}{\binom{N}{r}}.$$

In a 2×2-table the symbols m, n, r and s may be assigned to the marginal totals in such a way, that

$$(1;4) \qquad m \leqq n,\ r \leqq s \text{ and } r \leqq m.$$

Then, not considering the trivial case when $r = 0$, we have

$$(1;5) \qquad 0 < r \leqq m \leqq n \leqq s,$$

and the tailprobability of the 2×2-table may be found from the distribution of $\underline{a}_1$. Consequently for the purpose of approximating to the tailprobability it is sufficient to know the limit-distribution of $\underline{a}_1$ under the condition (1;5). The form of this limitdistribution (if it exists) depends on the asymptotic behaviour of the mean $\mathscr{E}\,\underline{a}_1 = mr/N$ and the variance $\sigma^2 = \sigma^2\,(\underline{a}_1) = mnrs/N^2(N-1)$; hence the approximation to be used depends on the mean $\mathscr{E}\underline{a}_1$ and the variance σ^2 found from the marginal totals realized in the experiment.

The possible forms of the limit-distribution (if it exists) of $\underline{a}_1$ under the condition (1;5) are: a univalued, a binomial, a Poisson and a normal distribution. These results are known, but in the literature on this subject we could find neither an exact proof nor a clear statement of the conditions to be imposed on $\mathscr{E}\,\underline{a}_1$ and σ^2. Therefore these conditions (and their practical interpretation for the approximationproblem) are summarized in section 2; the proofs are given in section 3.

In some cases the distribution of $\underline{a}_1$ does not have a limit, these cases are treated in section 4.

2. Limiting-distributions and conditions

Consider, for $\nu = 1, 2, \ldots$, the sequence of 2×2-tables

$$\begin{array}{cc|c} \underline{a}_{1,\nu} & \underline{a}_{3,\nu} & m_\nu \\ \underline{a}_{2,\nu} & \underline{a}_{4,\nu} & n_\nu \\ \hline r_\nu & s_\nu & N_\nu, \end{array}$$

where, for each ν,

$$\text{(2;1)} \qquad \mathrm{P}\,[\underline{a}_{1,\nu} = a] = \frac{\binom{m_\nu}{a}\binom{n_\nu}{r_\nu - a}}{\binom{N_\nu}{r_\nu}}.$$

Now let $\lim_{\nu \to \infty} N_\nu = \infty$ and consider the limit-distribution (under suitable normalization) of $\underline{a}_{1,\nu}$ for $\nu \to \infty$ under the condition (cf. (1;5))

$$\text{(2;2)} \qquad r_\nu \leqq m_\nu \leqq n_\nu \leqq s_\nu \text{ for each } \nu.$$

In order to simplify the notation the index ν will henceforth be omitted.

Let

$$\text{(2;3)} \qquad \begin{cases} \mu_i \overset{\text{def}}{=} \mathscr{E}\,\underline{a}_i \\ \sigma^2 \overset{\text{def}}{=} \sigma^2\,(\underline{a}_i) \end{cases} \qquad (i = 1, 2, 3, 4)$$

then

$$\text{(2;4)} \qquad \begin{cases} \mu_1 = \dfrac{mr}{N}, & \mu_3 = \dfrac{ms}{N} \\ \mu_2 = \dfrac{nr}{N}, & \mu_4 = \dfrac{ns}{N} \end{cases} \qquad \sigma^2 = \frac{mnrs}{N^2\,(N-1)}.$$

Further

$$\text{(2;5)} \qquad \begin{cases} \mu_1 + \mu_2 = r, \quad \mu_1 + \mu_3 = m \\ \mu_3 + \mu_4 = s, \quad \mu_2 + \mu_4 = n \\ \sum_{i=1}^{4} \mu_i = 2N \end{cases}$$

and

$$\text{(2;6)} \qquad \frac{N}{(N-1)\,\sigma^2} = \sum_{i=1}^{4} \frac{1}{\mu_i}.$$

From (2;2), (2;4) and (2;6) it follows that

$$\frac{1}{4}\frac{N}{N-1}\mu_1 \leqq \sigma^2 \leqq \mu_1 \leqq \mu_2 \leqq \mu_3 \leqq \mu_4 \tag{2;7}$$

and from (2;5) and (2;7) it follows that

$$\lim_{\nu\to\infty} \mu_4 = \infty. \tag{2;8}$$

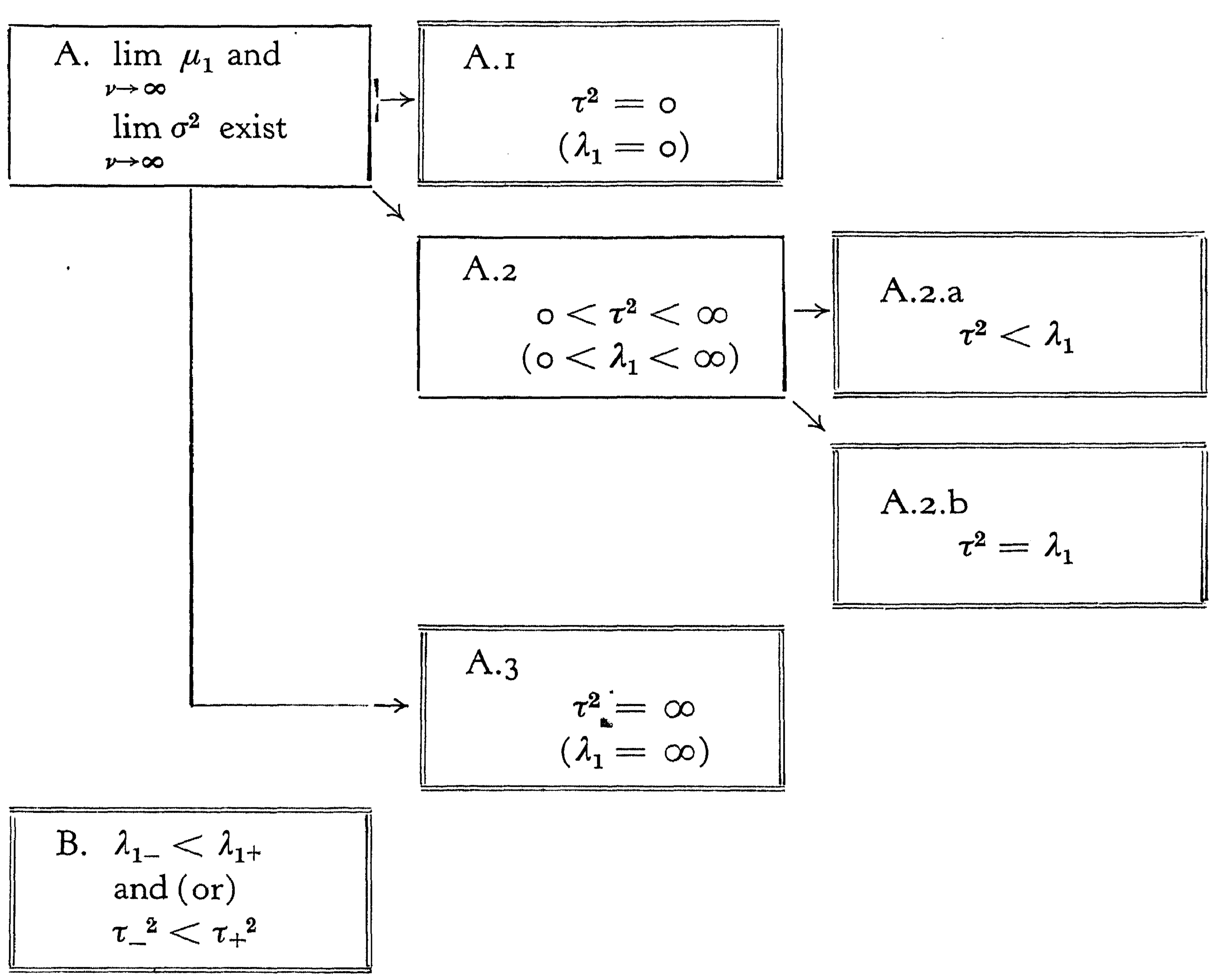
Let further (infinite values being allowed)

$$\begin{cases} \lambda_{1-} \overset{\text{def}}{=} \liminf_{\nu\to\infty} \mu_1, \ \tau_-^2 \overset{\text{def}}{=} \liminf_{\nu\to\infty} \sigma^2 \\ \lambda_{1+} \overset{\text{def}}{=} \limsup_{\nu\to\infty} \mu_1, \ \tau_+^2 \overset{\text{def}}{=} \limsup_{\nu\to\infty} \sigma^2. \end{cases} \tag{2;9}$$

If $\lambda_{1-} = \lambda_{1+}$ (or $\tau^2_- = \tau_+^2$) we say that $\lim_{\nu\to\infty} \mu_1$ (or $\lim_{\nu\to\infty} \sigma^2$) exists and denote it by λ_1 (or τ^2). Further (cf. (2;7))

$$\left.\begin{cases} \tau^2 = 0 \ \text{is equivalent with } \lambda_1 = 0, \\ 0 < \tau^2 < \infty \ \text{is equivalent with } 0 < \lambda_1 < \infty, \\ \tau^2 = \infty \ \text{is equivalent with } \lambda_1 = \infty. \end{cases}\right\} \text{if } \lim_{\nu\to\infty} \mu_1 \text{ and } \lim_{\nu\to\infty} \sigma^2 \text{ both exist} \tag{2;10}$$

Now the following cases may be distinguished



The cases to be considered are

A.1, A.2.a, A.2.b, A3 and B.

In case B the distribution of $\underline{a}_1$ does not have a limit. This will be proved in section 4.
In this section we consider the cases

A.1, A.2.a, A.2.b and A.3,

where $\underline{a}_1$ has a limit-distribution. Then the following theorems hold:

Theorem 1 (Case A. 1)
If $\tau^2 = 0$ ($\lambda_1 = 0$) then $\underline{a}_1$ has asymptotically a degenerate distribution

(2;11) $$\lim_{\nu\to\infty} \mathrm{P}\,[\underline{a}_1 = 0] = 1.$$

Theorem 2 (Case A.2.a)
If $0 < \tau^2 < \lambda_1 < \infty$ then $\underline{a}_1$ has asymptotically a non-degenerate binomial distribution with expectation λ_1 and variance τ^2:

(2;12) $$\lim_{\nu\to\infty} \mathrm{P}\,[\underline{a}_1 = a] = \binom{k}{a}\,\theta^a\,(1-\theta)^{k-a}\;(a = 0, 1, \ldots, k),$$

where

(2;13) $$k = \lim_{\nu\to\infty} r = \frac{\lambda^2{}_1}{\lambda_1 - \tau^2},\quad \theta = \lim_{\nu\to\infty}\frac{m}{N} = 1 - \frac{\tau^2}{\lambda_1}.$$

Remark: In this case $\underline{a}_2$ also has asymptotically a binomial distribution

(2.14) $$\lim_{\nu\to\infty} \mathrm{P}\,[\underline{a}_2 = a] = \binom{k}{a}\,(1-\theta)^a\,\theta^{k-a}\;(a = 0, 1, \ldots, k).$$

Further μ_3 and μ_4 tend to infinity with ν.

Theorem 3 (Case A.2.b)
If $0 < \tau^2 = \lambda_1 < \infty$ then $\underline{a}_1$ has asymptotically a non-degenerate Poisson distribution

(2;15) $$\lim_{\nu\to\infty} \mathrm{P}\,[\underline{a}_1 = a] = \frac{e^{-\lambda_1}\,\lambda_1{}^a}{a!}\;(a = 0, 1, \ldots).$$

Remark: In this case $\lim_{\nu\to\infty} \mu_i = \infty$ for $i = 2$, 3 and 4. Further all marginal totals tend to infinity with ν.

Theorem 4 (Case A.3)

If $\tau^2 = \infty\,(= \lambda_1)$ then all random variables $\frac{\underline{a}_i - \mu_i}{\sigma}$ have asymptotically a N (0,1)-distribution, i.e.

$$(2;16) \qquad \lim_{\nu \to \infty} \mathrm{P}\left[u_1 \leqq \frac{\underline{a}_i - \mu_i}{\sigma} \leqq u_2\right] = \frac{1}{\sqrt{2\pi}} \int_{u_1}^{u_2} e^{-\frac{1}{2}u^2}\, du$$

for any finite u_1 and u_2 ($u_1 < u_2$).

Remark: In this case all μ_i and all marginal totals tend to infinity with ν.

These theorems will be proved in section 3.

Remarks on application

Consider a 2 × 2-table with a large value of N and suppose one wants to approximate to its tailprobability. Then the abovementioned theorems may be applied as follows:

The symbols m, n, r and s are assigned to the marginal totals of the 2 × 2-table in such a way that (cf. (2;2)).

$$(2;17) \qquad r \leqq m \leqq n \leqq s,$$

then (cf. (2;7))

$$(2;18) \qquad \mu_1 \leqq \mu_2 \leqq \mu_3 \leqq \mu_4.$$

In each particular case one has to decide which of the following alternatives best fits the situation on hand:

1. μ_1 is very small (say $\mu_1 \ll 1$). Then according to theorem 1

$$(2;19) \qquad \mathrm{P}\,[\underline{a}_1 = 0] \approx 1, \quad \mathrm{P}\,[\underline{a}_1 \geqq 1] \approx 0.$$

However, in this case a more useful relation may be obtained by using the inequality (cf. the proof of theorem 1)

$$(2;20) \qquad \mathrm{P}\,[\underline{a}_1 \geqq 1] \leqq \mu_1.$$

2. μ_1 and μ_2 are small, μ_3 and μ_4 are large. Then m, n and s are large, r is small and $\underline{a}_1$ has approximately a binomial distribution with parameters r and $\frac{m}{N}$, i.e.

$$(2;21) \qquad \mathrm{P}\,[\underline{a}_1 = a] \approx \binom{r}{a} \left(\frac{m}{N}\right)^a \left(\frac{n}{N}\right)^{r-a} \quad (a = 0, 1, \ldots, r).$$

This binomial approximation to the hypergeometric distribution is e.g.

mentioned by H. G. R o m i g (1953) in the introduction of his table of the binomial distribution. However, he does not mention all possible situations in which this approximation may be used (cf. also J. H e m e l r ij k (1954) in his review of R o m i g's table).

W. F e l l e r (1957, p. 57) also mentions this approximation. He gives the inequalities

(2;22)

$$\binom{r}{a}\left(\frac{m}{N}-\frac{a}{N}\right)^{a}\left(\frac{n}{N}-\frac{r-a}{N}\right)^{r-a} < \mathrm{P}\,[\underline{a}_1 = a] < \binom{r}{a}\left(\frac{m}{N}\right)^{a}\left(\frac{n}{N}\right)^{r-a}\left(\frac{s}{N}\right)^{-r}.$$

3. μ_1 is small, μ_2, μ_3 and μ_4 are large. Then all marginal totals are large and $\underline{a}_1$ has approximately a Poisson distribution with parameter μ_1

(2;23) $$\mathrm{P}\,[\underline{a}_1 = a] \approx \frac{e^{-\mu_1}\,{\mu_1}^{a}}{a!}\,(a = 0, 1, \ldots).$$

4. All μ_i are large. Then all marginal totals are large and the random variable $\underline{a}_1$ has approximately a normal distribution with mean μ_1 and variance σ^2

(2;24) $$\mathrm{P}\left[\frac{\underline{a}_1-\mu_1}{\sigma} \leqq u_1\right] \approx \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{u_1} e^{-\frac{1}{2}u^2}\,du.$$

3. Proof of the theorems of section 2

3.1. Proof of theorem 1

We have

(3.1;1) $$1 \geqq \mathrm{P}\,[\underline{a}_1 = 0] = 1 - \sum_{a\geqq 1}\mathrm{P}\,[\underline{a} = a] \geqq 1 - \sum_{a\geqq 1} a\,\mathrm{P}\,[\underline{a} = a] = 1 - \mu_1.$$

From $\lambda_1 = 0$ then follows

(3.1;2) $$\lim_{\nu\to\infty}\mathrm{P}\,[\underline{a}_1 = 0] = 1.$$

3.2. Proof of theorem 2

From $\lambda_1 = \lim\limits_{\nu\to\infty}\frac{mr}{N} > 0$ it follows that m tends to infinity with ν and from $m = \mu_1 + \mu_3$ and $\lambda_1 < \infty$ it follows that μ_3 tends to infinity with ν. Further (cf. (2;6))

(3.2;1) $$\lim_{\nu\to\infty}\sum_{i=1}^{4}\frac{\sigma^2}{\mu_i} = 1.$$

Consequently, as μ_3 and μ_4 tend to infinity with ν and τ^2 is finite, we have

$$(3.2;2) \qquad \lim_{\nu\to\infty} \sum_{i=1}^{2} \frac{\sigma^2}{\mu_i} = 1.$$

From (3.2;2), the fact that $\lim\limits_{\nu\to\infty} \frac{\sigma^2}{\mu_1}$ exists and $0 < \frac{\tau^2}{\lambda_1} < 1$ it follows that $\lim\limits_{\nu\to\infty} \frac{\sigma^2}{\mu_2}$ exists and $0 < \lim\limits_{\nu\to\infty} \frac{\sigma^2}{\mu_2} < 1$. Consequently $\lambda_2 \overset{\text{def}}{=} \lim\limits_{\nu\to\infty} \mu_2$ exists and $0 < \lambda_2 < \infty$. From $r = \mu_1 + \mu_2$ then follows that r tends to a finite positive limit; r being an integer this means that from a certain value of ν onwards r remains constant. From this and from $\lambda_1 = \lim\limits_{\nu\to\infty} \frac{mr}{N}$ and $\lambda_2 = \lim\limits_{\nu\to\infty} \frac{nr}{N}$ it follows that $\lim\limits_{\to\infty} \frac{m}{N}$ and $\lim\limits_{\nu\to\infty} \frac{n}{N}$ exist and $0 < \lim\limits_{\nu\to\infty} \frac{m}{N} \leqq \lim\limits_{\nu\to\infty} \frac{n}{N} < 1$.

According to (1;2) we have

$$(3.2;3) \quad \mathrm{P}\,[\underline{a}_1 = a] = \frac{\binom{r}{a}\binom{s}{m-a}}{\binom{N}{m}} = \binom{r}{a} \frac{\prod\limits_{j=1}^{a} (m-j+1) \prod\limits_{j=1}^{r-a} (n-j+1)}{\prod\limits_{j=1}^{r} (N-j+1)} =$$

$$= \binom{r}{a} \left(\frac{m}{N}\right)^a \left(\frac{n}{N}\right)^{r-a} \frac{\prod\limits_{j=1}^{a} \left(1 - \frac{j-1}{m}\right) \prod\limits_{j=1}^{r-a} \left(1 - \frac{j-1}{n}\right)}{\prod\limits_{j=1}^{r} \left(1 - \frac{j-1}{N}\right)}.$$

Further, r having a finite limit and m, n and N tending to infinity with ν,

$$(3.2;4) \lim_{\nu\to\infty} \prod_{j=1}^{a} \left(1 - \frac{j-1}{m}\right) = \lim_{\nu\to\infty} \prod_{j=1}^{r-a} \left(1 - \frac{j-1}{n}\right) = \lim_{\nu\to\infty} \prod_{j=1}^{r} \left(1 - \frac{j-1}{N}\right) = 1.$$

Consequently

$$(3.2;4) \quad \lim_{\nu\to\infty} \mathrm{P}\,[\underline{a}_1 = a] = \lim_{\nu\to\infty} \binom{r}{a} \left(\frac{m}{N}\right)^a \left(\frac{n}{N}\right)^{r-a} = \binom{k}{a} \theta^a (1-\theta)^{k-a},$$

with

$$(3.2;5) \qquad k = \lim_{\nu\to\infty} r, \ \theta = \lim_{\nu\to\infty} \frac{m}{N}.$$

Further, r being equal to k for sufficiently large ν, $\underline{a}_1$ can only take the $k + 1$ values $0, 1, \ldots, k$ and the limits of μ_1 and σ^2 are equal to the corresponding moments of the limit-distribution. Thus

(3.2;6) $$\lambda_1 = k\theta \text{ and } \tau^2 = k\theta(1 - \theta),$$

consequently

(3.2;7) $$k = \frac{\lambda_1^2}{\lambda_1 - \tau^2}, \quad \theta = 1 - \frac{\tau^2}{\lambda_1}.$$

3.3. **Proof of theorem 3**

From

(3.3;1) $$\lim_{\nu\to\infty} \sum_{i=1}^{4} \frac{\sigma^2}{\mu_i} = 1$$

and $0 < \tau^2 = \lambda_1 < \infty$ it follows that $\lim_{\nu\to\infty} \mu_i = \infty$ for $i = 2$, 3 and 4. From (2;5) it then follows that all marginal totals tend to infinity with ν. Further $\lambda_1 = \lim_{\nu\to\infty} \frac{mr}{N} < \infty$; consequently

(3.3;2) $$\lim_{\nu\to\infty} \frac{m}{N} = 0, \quad \lim_{\nu\to\infty} \frac{r}{N} = 0.$$

From (1;2) it follows that

(3.3;3) $$\mathrm{P}\,[\underline{a}_1 = a] = \frac{1}{a!} \frac{\prod_{j=1}^{a} (m - j + 1) \prod_{j=1}^{a} (r - j + 1) \prod_{j=1}^{r-a} (n - j + 1)}{\prod_{j=1}^{r} (N - j + 1)} =$$

$$= \frac{1}{a!} \left(\frac{mr}{N}\right)^a \frac{\prod_{j=1}^{a} \left(1 - \frac{j-1}{m}\right) \prod_{j=1}^{a} \left(1 - \frac{j-1}{r}\right)}{\prod_{j=r-a+1}^{r} \left(1 - \frac{j-1}{N}\right)} \prod_{j=1}^{r-a} \frac{n-j+1}{N-j+1}.$$

Now we have, for each finite a, all marginal totals tending to infinity with ν,

(3.3;4) $$\lim_{\nu\to\infty} \prod_{j=1}^{a} \left(1 - \frac{j-1}{m}\right) = \lim_{\nu\to\infty} \prod_{j=1}^{a} \left(1 - \frac{j-1}{r}\right) = 1.$$

Further, $\frac{r}{N}$ tending to zero for $\nu \to \infty$,

(3.3;5) $$\lim_{\nu\to\infty} \prod_{j=r-a+1}^{r} \left(1 - \frac{j-1}{N}\right) = \lim_{\nu\to\infty} \prod_{j=1}^{a} \left(1 - \frac{r-a+j-1}{N}\right) = 1.$$

Consequently

$$(3.3;6)\qquad \lim_{\nu\to\infty} \mathrm{P}\,[\underline{a}_1 = a] = \frac{1}{a!}\lambda_1^a \lim_{\nu\to\infty} \prod_{j=1}^{r-a} \frac{n-j+1}{N-j+1} = \frac{1}{a!}\lambda_1^a \lim_{\nu\to\infty} \prod_{j=1}^{r} \frac{n-j+1}{N-j+1}$$

and there remains to prove that

$$(3.3;7)\qquad \lim_{\nu\to\infty} \prod_{j=1}^{r} \frac{n-j+1}{N-j+1} = e^{-\lambda}$$

or

$$(3.3;8)\quad \lim_{\nu\to\infty} \sum_{j=1}^{r} \ln \frac{n-j+1}{N-j+1} = \lim_{\nu\to\infty} \sum_{j=1}^{r} \ln\left(1 - \frac{m}{N-j+1}\right) = -\lambda.$$

Now we have

$$(3.3;9)\quad r \ln\left(1-\frac{m}{N-r+1}\right) \leqq \sum_{j=1}^{r} \ln\left(1-\frac{m}{N-j+1}\right) \leqq r\ln\left(1-\frac{m}{N}\right).$$

Further, $\frac{m}{N}$ and $\frac{r}{N}$ tending to zero with ν, we have

$$(3.3;10)\ \lim_{\nu\to\infty} r\ln\left(1-\frac{m}{N-r+1}\right) = \lim_{\nu\to\infty} r\ln\left(1-\frac{m}{N}\right) = \lim_{\nu\to\infty}\frac{mr}{N} = -\lambda$$

and (3.3;8) follows from (3.3;9) and (3.3;10).

3.4. **Proof of theorem 4**

From $\mu_1 \leqq \mu_2 \leqq \mu_3 \leqq \mu_4$ and $\lambda_1 = \infty$ it follows that all μ_i and all marginal totals tend to infinity.

The proof of the asymptotic normality of the distribution of $\frac{\underline{a}_i - \mu_i}{\sigma}$ is analogous to the proof given by W. Feller (1957, p. 168—173) for the asymptotic normality of the binomial distribution; i.e. we use Stirling's formula for $\Gamma(p+1)$

$$(3.4;1)\quad \Gamma(p+1) = \left(\frac{p}{e}\right)^p \sqrt{2\pi p}\,\exp\left\{O\left(\frac{1}{p}\right)\right\} \quad \text{where} \left|O\left(\frac{1}{p}\right)\right| \leqq \frac{1}{6p}.$$

The proof will be given for $i = 1$. The asymptotic normality of $\frac{\underline{a}_i - \mu_i}{\sigma}$ for $i = 2$, 3 and 4 then follows from the fact that

$$(3.4;2)\qquad \underline{a}_1 - \mu_1 = -(\underline{a}_2 - \mu_2) = -(\underline{a}_3 - \mu_3) = \underline{a}_4 - \mu_4.$$

Now we have (cf. (1;2))

$$(3.4;3)\quad P[\underline{a}_1 = a] = \frac{m!n!r!s!}{N!a!\,(m-a)!(r-a)!(n-r+a)!} =$$
$$= \frac{m!n!r!s!}{N!\,\Gamma(\mu_1+1)\,\Gamma(\mu_2+1)\,\Gamma(\mu_3+1)\,\Gamma(\mu_4+1)}$$
$$\frac{\Gamma(\mu_1+1)\,\Gamma(\mu_2+1)\,\Gamma(\mu_3+1)\,\Gamma(\mu_4+1)}{a!\,(m-a)!\,(r-a)!\,(n-r+a)!}.$$

Further (cf. (3.4;1))

$$(3.4;4)\quad \frac{m!n!r!s!}{N!\,\Gamma(\mu_1+1)\,\Gamma(\mu_2+1)\,\Gamma(\mu_3+1)\,\Gamma(\mu_4+1)} =$$
$$= \frac{1}{\sigma\sqrt{2\pi}}\sqrt{\frac{N}{N-1}}\exp\left\{O_1\left(\frac{1}{\sigma}\right)\right\},$$

where — as can easily be seen — :

(3.4;5)
$$\left|O_1\left(\frac{1}{\sigma}\right)\right| \leqq \frac{1}{6}\left(\frac{1}{m}+\frac{1}{n}+\frac{1}{r}+\frac{1}{s}+\frac{1}{N}+\frac{1}{\mu_1}+\frac{1}{\mu_2}+\frac{1}{\mu_3}+\frac{1}{\mu_4}\right) \leqq \frac{1}{2\sigma}\sum_{i=1}^{4}\frac{\sigma}{\mu_i}\text{[1]}.$$

Now we have

$$(3.4;6)\qquad \frac{\sigma}{\mu_i} \leqq \frac{1}{\sqrt{\mu_i}} \quad (i = 1,\ 2,\ 3,\ 4),$$

consequently (μ_i tending to infinity with ν for each $i = 1, 2, 3, 4$) for each positive δ a $\nu(\delta)$ exists such that, for $\nu > \nu(\delta)$,

$$(3.4;7)\qquad \frac{\sigma}{\mu_i} \leqq \frac{1}{\sqrt{\mu_i}} \leqq \delta \quad \text{for each } i = 1,\ 2,\ 3,\ 4.$$

Hence, for $\nu > \nu(\delta)$, we have

$$(3.4;8)\qquad \left|O_1\left(\frac{1}{\sigma}\right)\right| \leqq \frac{2\delta}{\sigma}.$$

Now let

$$(3.4;9)\qquad x \overset{\text{def}}{=} \frac{a-\mu_1}{\sigma},$$

then

[1]) The relation even holds with the <-sign. For simplicity we use ≦ everywhere in this proof.

$$(3.4;10)\quad \frac{\Gamma(\mu_1+1)\,\Gamma(\mu_2+1)\,\Gamma(\mu_3+1)\,\Gamma(\mu_4+1)}{a!\,(m-a)!\,(r-a)!\,(n-r+a)!}=$$

$$=\frac{\mu_1^{\mu_1+\frac{1}{2}}\,\mu_2^{\mu_2+\frac{1}{2}}\,\mu_3^{\mu_3+\frac{1}{2}}\,\mu_4^{\mu_4+\frac{1}{2}}}{(\mu_1+x\sigma)^{\mu_1+x\sigma+\frac{1}{2}}\,(\mu_2-x\sigma)^{\mu_2-x\sigma+\frac{1}{2}}\,(\mu_3-x\sigma)^{\mu_3-x\sigma+\frac{1}{2}}\,(\mu_4+x\sigma)^{\mu_4+x\sigma+\frac{1}{2}}}\,.$$

$$.\exp\left\{O_2\left(\frac{1}{\sigma}\right)\right\},$$

where

$$(3.4;11)\quad \left|O_2\left(\frac{1}{\sigma}\right)\right|\leqq\frac{1}{6}\left\{\sum_{i=1}^{4}\frac{1}{\mu_i}+\frac{1}{|\mu_1+x\sigma|}+\frac{1}{|\mu_2-x\sigma|}+\frac{1}{|\mu_3-x\sigma|}+\frac{1}{|\mu_4+x\sigma|}\right\}.$$

Now let $|x|\leqq x_0$, where x_0 is a finite positive number; then, for $\nu>\nu(\delta)$,

$$(3.4;12)\qquad |x|\frac{\sigma}{\mu_i}\leqq x_0\,\delta\quad\text{for each } i=1,2,3,4.$$

Let further ε be a positive number $\leqq\frac{1}{3}$, then we choose δ in such a way that

$$(3.4;13)\qquad x_0\delta\leqq\varepsilon \text{ and } \delta\leqq\varepsilon.$$

Then we have

$$(3.4;14)\qquad \frac{\sigma}{|\mu_i\pm x\sigma|}=\frac{\dfrac{\sigma}{\mu_i}}{\left|1\pm x\dfrac{\sigma}{\mu_i}\right|}\leqq\frac{\delta}{1-\varepsilon}\leqq 2\,\varepsilon \text{ for each } i=1,2,3,4.$$

Consequently for $\nu>\nu(\delta)$ we have

$$(3.4;15)\qquad \begin{cases}\left|O_1\left(\dfrac{1}{\sigma}\right)\right|\leqq\dfrac{2\varepsilon}{\sigma}\\[2ex] \left|O_2\left(\dfrac{1}{\sigma}\right)\right|\leqq\dfrac{1}{6\sigma}\{4\varepsilon+8\varepsilon\}\leqq\dfrac{2\varepsilon}{\sigma}.\end{cases}$$

Further

$$(3.4;16)\qquad \ln\frac{\mu_i^{\mu_i+\frac{1}{2}}}{(\mu_i\pm x\sigma)^{\mu_i\pm x\sigma+\frac{1}{2}}}=$$

$$=\mp x\sigma\ln\mu_i-(\mu_i\pm x\sigma+\tfrac{1}{2})\ln\left(1\pm x\frac{\sigma}{\mu_i}\right)\quad(i=1,2,3,4).$$

Consequently the logarithm of the first factor in the righthand side of (3.4;10) equals

$$(3.4;17)\ -x\sigma\ln\frac{\mu_1\mu_4}{\mu_2\mu_3}-(\mu_1+x\sigma+\tfrac{1}{2})\ln\left(1+x\frac{\sigma}{\mu_1}\right)-(\mu_2-x\sigma+\tfrac{1}{2})\ln\left(1-x\frac{\sigma}{\mu_2}\right)+$$
$$-(\mu_3-x\sigma+\tfrac{1}{2})\ln\left(1-x\frac{\sigma}{\mu_3}\right)-(\mu_4+x\sigma+\tfrac{1}{2})\ln\left(1+x\frac{\sigma}{\mu_4}\right),$$

where

$$(3.4;18)\qquad \ln\frac{\mu_1\mu_4}{\mu_2\mu_3}=0.$$

Now we have for $|\ u\ |\leqq\frac{1}{3}$

$$(3.4;19)\qquad \ln(1+u)=u-\frac{u^2}{2}+O(u^3),$$

where

$$(3.4;20)\ |O(u^3)|=\left|\ln(1+u)-u+\frac{u^2}{2}\right|=\left|-\sum_{i=3}^{\infty}\frac{(-u)^i}{i}\right|\leqq\frac{|u|^3}{3}\sum_{i=3}^{\infty}(\tfrac{1}{3})^{i-3}=\tfrac{1}{2}|u|^3.$$

Using (3.4;19) with $u=\pm x\frac{\sigma}{\mu_i}$ we find that, for $\nu>\nu\ (\delta)$ and $\varepsilon\leqq\frac{1}{3}$, (3.4;17) equals

$$(3.4;21)\qquad -\tfrac{1}{2}x^2\frac{N}{N-1}+O_3\left(\frac{1}{\sigma}\right),$$

where

$$(3.4;22)\ \left|O_3\left(\frac{1}{\sigma}\right)\right|\leqq\frac{1}{\sigma}\left\{\tfrac{1}{2}x_0\frac{N}{N-1}+x_0^3\sum_{i=1}^{4}\frac{\sigma^4}{\mu_i^2}+\tfrac{1}{4}x_0^2\sum_{i=1}^{4}\frac{\sigma^3}{\mu_i^2}+\right.$$
$$\left.+\tfrac{1}{2}x_0^4\sum_{i=1}^{4}\frac{\sigma^5}{\mu_i^3}+\tfrac{1}{4}x_0^3\sum_{i=1}^{4}\frac{\sigma^4}{\mu_i^3}\right\}.$$

Further we have

$$(3.4;23)\qquad \sum_{i=1}^{4}\frac{\sigma^4}{\mu_i^2}\leqq\left(\sum_{i=1}^{4}\frac{\sigma^2}{\mu_i}\right)^2=\left(\frac{N}{N-1}\right)^2\leqq 4$$

and, for $\nu>\nu\ (\delta)$,

$$(3.4;24)\ \begin{cases} x_0^2\sum_{i=1}^{4}\frac{\sigma^3}{\mu_i^2}=x_0\sum_{i=1}^{4}\frac{\sigma^2}{\mu_i}\cdot x_0\frac{\sigma}{\mu_i}\leqq x_0\,\varepsilon\frac{N}{N-1}\leqq 2x_0\,\varepsilon,\\ x_0^4\sum_{i=1}^{4}\frac{\sigma^5}{\mu_i^3}=x_0^3\sum_{i=1}^{4}\frac{\sigma^4}{\mu_i^2}\cdot x_0\frac{\sigma}{\mu_i}\leqq 4x_0^3\,\varepsilon,\\ x_0^3\sum_{i=1}^{4}\frac{\sigma^4}{\mu_i^3}=x_0^2\sum_{i=1}^{4}\frac{\sigma_4}{\mu_i^2}\frac{x_0}{\mu_i}\leqq 4x_0^2\,\varepsilon^2.\end{cases}$$

Consequently, for $\nu>\nu\ (\delta)$ and $\varepsilon\leqq\frac{1}{3}$,

$$(3.4;25)\qquad \left|O_3\left(\frac{1}{\sigma}\right)\right|\leqq\frac{1}{\sigma}\{2x_0^3(2+\varepsilon)+x_0^2\varepsilon^2+\tfrac{1}{2}x_0(2+\varepsilon)\}.$$

Substituting these results in (3.4;3) we obtain, for $\frac{|a-\mu_1|}{\sigma} \leqq x_0$, $\nu > \nu(\delta)$ and $\varepsilon \leqq \frac{1}{3}$,

$$(3.4;26)\quad P[\underline{a}=a] = \frac{1}{\sigma\sqrt{2\pi}} \sqrt{\frac{N}{N-1}} \exp\left\{O_4\left(\frac{1}{\sigma}\right)\right\} \exp\left\{-\frac{1}{2}\left(\frac{a-\mu_1}{\sigma}\right)^2 \frac{N}{N-1}\right\},$$

where

$$(3.4;27)\qquad \left|O_4\left(\frac{1}{\sigma}\right)\right| \leqq \left|O_1\left(\frac{1}{\sigma}\right) + O_2\left(\frac{1}{\sigma}\right) + O_3\left(\frac{1}{\sigma}\right)\right| \leqq$$

$$\leqq \frac{1}{\sigma}\{4\varepsilon + 2x_0^3(2+\varepsilon) + x_0^2\varepsilon^2 + \tfrac{1}{2}x_0(2+\varepsilon)\}.$$

From (3.4;26) then follows that, for $x_1 < x_2$, $|x_1| \leqq x_0$, $|x_2| \leqq x_0$

$$(3.4;28)\qquad P\left[x_1 < \frac{\underline{a}_1-\mu_1}{\sigma} \leqq x_2\right] = \sum_{a=[\mu_1+x_1\sigma+1]}^{[\mu_1+x_2\sigma]} P[\underline{a}=a] =$$

$$= \frac{1}{\sqrt{2\pi}} \sqrt{\frac{N}{N-1}} \exp\left\{O_4\left(\frac{1}{\sigma}\right)\right\} \sum_{a=[\mu_1+x_1\sigma+1]}^{[\mu_1+x_2\sigma]} \frac{1}{\sigma} \exp\left\{-\frac{1}{2}\left(\frac{a-\mu_1}{\sigma}\right)^2 \frac{N}{N-1}\right\},$$

where

$$\sum_{a=[\mu_1+x_1\sigma+1]}^{[\mu_1+x_2\sigma]} \frac{1}{\sigma} \exp\left\{-\frac{1}{2}\left(\frac{a-\mu_1}{\sigma}\right)^2 \frac{N}{N-1}\right\}$$

is a Riemann-sum approximating the integral $\int_{x_1}^{x_2} e^{-\frac{1}{2}x^2}\,dx$. Hence we proved that for any finite x_1 and x_2 with $x_1 < x_2$

$$(3.4;29)\qquad \lim_{\nu\to\infty} P\left[x_1 \leqq \frac{\underline{a}_1-\mu_1}{\sigma} \leqq x_2\right] = \frac{1}{\sqrt{2\pi}} \int_{x_1}^{x_2} e^{-\frac{1}{2}x^2}\,dx.$$

Remark

A proof of the asymptotic normality of the distribution of $\frac{\underline{a}_1-\mu_1}{\sigma}$ under the more stringent conditions (1;5) and

$$(3.4;30)\qquad \begin{cases} 1.\ r \text{ tends to infinity with } \nu, \\ 2.\ \liminf_{\nu\to\infty} \frac{m}{N} > 0 \end{cases}$$

has been given by G. Madow (1948).

4. The cases where the distribution of $\underline{a}_1$ does not have a limit

In this section we consider case B of section 2. It will be proved that in this case the distribution of $\underline{a}_1$ does not have a limit. In case B we have: at least one of the limits $\lim\limits_{\nu\to\infty} \mu_1$ and $\lim\limits_{\nu\to\infty} \sigma^2$ does not exist and the following cases may be distinguished.

I. $\lim\limits_{\nu\to\infty} \sigma^2$ exists, then (cf (2;7) and (2;10))

$$0 < \tau^2 \leqq \lambda_{1-} < \lambda_{1+} < \infty. \tag{4;1}$$

Then two subsequences $\{\nu'\}$ and $\{\nu''\}$ of the sequence $\{\nu\}$ exist with

$$\begin{cases} \lim\limits_{\nu'\to\infty} N = \infty, \ \lim\limits_{\nu''\to\infty} N = \infty, \\ \lim\limits_{\nu'\to\infty} \mu_1 = \lambda_{1-}, \ \lim\limits_{\nu''\to\infty} \mu_1 = \lambda_{1+}. \end{cases} \tag{4;2}$$

From theorem 2 it then follows that $\underline{a}_1$ has asymptotically for $\nu'' \to \infty$ a binomial distribution with mean λ_{1+} and variance τ^2. Further, if $\tau^2 < \lambda_{1-}$, $\underline{a}_1$ has asymptotically for $\nu' \to \infty$ a binomial distribution with mean λ_{1-} and variance τ^2; if $\tau^2 = \lambda_{1-}$ it follows from theorem 3 that $\underline{a}_1$ has asymptotically for $\nu' \to \infty$ a Poisson distribution with parameter λ_{1-}. Consequently the distributions of $\underline{a}_1$ for $\nu' \to \infty$ and for $\nu'' \to \infty$ are not identical; i.e. $\underline{a}_1$ does not have a limit distribution for $\nu \to \infty$.

II. $\lim\limits_{\nu\to\infty} \sigma^2$ does not exist. Then (cf (2;7) and (2;10))

$$\begin{cases} 0 < \tau_+^2 \leqq \lambda_{1+} \ , \ \tau_-^2 \leqq \lambda_{1-} < \infty, \\ \tau_-^2 < \tau_+^2 \ , \ \lambda_{1-} \leqq \lambda_{1+}. \end{cases} \tag{4;3}$$

Then two subsequences $\{\nu'\}$ and $\{\nu''\}$ of the sequence $\{\nu\}$ exist with

$$\begin{cases} \lim\limits_{\nu'\to\infty} N = \infty \ , \ \lim\limits_{\nu''\to\infty} N = \infty, \\ \lim\limits_{\nu'\to\infty} \sigma^2 = \tau^2_- \ , \ \lim\limits_{\nu''\to\infty} \sigma^2 = \tau_+^2 \end{cases} \tag{4;4}$$

and the following two cases may be distinguished.

1. at least one of the limits $\lim\limits_{\nu'\to\infty} \mu_1$ and $\lim\limits_{\nu''\to\infty} \mu_1$ does not exist. Then it follows from the foregoing that $\underline{a}_1$ does not have a limit distribution for $\nu' \to \infty$ and (or) for $\nu'' \to \infty$. Consequently in this case $\underline{a}_1$ does not have a limit distribution for $\nu \to \infty$.

2. $\lambda_1' \overset{\text{def}}{=} \lim\limits_{\nu' \to \infty} \mu_1$ and $\lambda_1'' \overset{\text{def}}{=} \lim\limits_{\nu'' \to \infty} \mu_1$ exist. Then (cf (4;3))

$$(4;5) \qquad \begin{cases} \tau_-^2 \leqq \lambda_1' < \infty, \quad 0 < \tau_+^2 \leqq \lambda_1'' \\ \tau_-^2 < \tau_+^2 \end{cases}$$

and $\underline{a}_1$ has a limit distribution for $\nu' \to \infty$ and for $\nu'' \to \infty$. The limit distribution of $\underline{a}_1$ for $\nu' \to \infty$ is

a. a degenerate distribution if $\tau_-^2 = 0$ (then also $\lambda_1' = 0$),
b. a non-degenerate binomial distribution with mean λ_1' and variance τ_-^2 if $\tau_-^2 < \lambda_1'$,
c. a non-degenerate Poisson-distribution with parameter λ_1', if $\tau_-^2 = \lambda_1' > 0$.

The limit-distribution of $\underline{a}_1$ for $\nu'' \to \infty$ is

a. a non-degenerate binomial distribution with mean λ_1'' and variance τ_+^2 if $\tau_+^2 < \lambda_1''$,
b. a non-degenerate Poisson-distribution with parameter λ_1'' if $\tau_+^2 = \lambda_1'' < \infty$,
c. after standardization a normal distribution if $\tau_+^2 = \lambda_1'' = \infty$. From $\tau_-^2 < \tau_+^2$ it then follows that the distributions of $\underline{a}_1$ for $\nu' \to \infty$ and for $\nu'' \to \infty$ are not identical.

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